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The Rigged Hilbert Space and Quantum Mechanics

by

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PREFACE

The purpose of this paper is not to present new results, but to make a little-known area in the mathematical foundations of quantum mechanics accessable to a wider audience.* Therefore the discussion of the physics and mathematics will be limited to the bare essentials. The whole subject is developed in terms of the simple, well-known example of the harmonic oscillator, and the mathematical notions have been introduced in the least general form which is compatible with the purpose of this paper (Unfortunately, the mathematics had to be abbreviated to an extent which, in the second part, precludes the possibility of proving the mathematical statements, but which, hopefully, still allows one to understand the content of these mathematical statements). The general mathematical background has been treated in several monographs[1,2] written by the pioneers of this area, to which we refer the interested reader. Remarks (ending with the symbol o) have been inserted throughout the paper to mention generalizations and additional related material.

One can argue that the subject described here is very useful for physics, because it makes the Dirac formalism rigorous and therewith gives a mathematical justification for all the mathematically undefined operations which

*It is based on a series of lectures given to mathematicians and physicists.

ERRATUM

to

"The Rigged Hilbert Space and Quantum Mechanics"

The fourth and fifth lines of Page 9, which read

> That is, X_i, $i = 1,2,..n$ are the generators of A, then the operator $\Delta = \sum_{i=1}^{n} X_i^2$ is essentially self-adjoint (e.s.a.).

should be replaced by the following:

> Also, the operator $\Delta = \sum_{i=1}^{n} X_i^2$, where X_i, $i = 1,2,..n$ are the generators of A, is essentially self-adjoint (e.s.a.).

physicists have been using for generations. It also gives a slightly different framework for quantum mechanics than the von Neumann axioms. But all this appears to me negligible compared to the main feature of this mathematical structure, its beauty. The Rigged Hilbert Space has been one of the most beautiful pieces of art I have come across and has given me joy whenever I looked at it. I hope that the simplified picture which is shown in this paper still conveys enough of this beauty to please even those physicists who avoid new mathematics because they see no need for complete mathematical rigour in physical calculations.

In the following introductory section a general framework for the formulation of quantum mechanics in the Rigged Hilbert Space is given. Though modifications of this framework which incorporate the same mathematical structure are possible, it is within this framework that the subject is developed in the succeeding sections.

In section I the algebraic representation space for the algebra generated by the momentum and position operator is constructed in a way very familiar to physicists, the essential assumption is that there exists at least one eigenvector of the energy operator.

In section II this algebraic space is equipped with two different topologies, the usual Hilbert space topology

and a stronger, nuclear topology. This nuclear topology has a dual purpose 1. the elements of the algebra are represented by continuous operators and 2. there exist eigenvectors of the momentum and position operators. (-as explained in the following sections-) neither of which is the case for the usual Hilbert space formulation.

In section III the spaces of antilinear functionals are described and the Rigged Hilbert Space is constructed.

Section IV gives the definition of generalized eigenvectors and a statement of the Nuclear Spectral Theorem, which guarantees the existence of a complete set of generalized eigenvectors of momentum and position operators. Realizations of the Rigged Hilbert Space by spaces of functions and distributions are described and in Appendix V a simple derivation of the Schrödinger realization is given.

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0. INTRODUCTION

To obtain a physical theory means to obtain a mathematical image of a physical system. The mathematical image for the domain of quantum phenomena is, according to von Neumann[3], the set of operators in a Hilbert space. For the case that no superselection rules are present (irreducible systems) the physical observables are identified with the set of all self-adjoint operators in the Hilbert space H and the ("pure") physical states are identified with the set of all unit rays, or equivalently, the one dimensional projection operators in H.

This one-to-one correspondence between the physical objects of a system and the operators in a Hilbert space has two deficiencies:

1) It does not contain all the information one has in quantum mechanics

2) It contains more information than one can ever obtain in physics.

To elaborate on 1.): It is well known that all (separable infinite dimensional) Hilbert spaces are isomorphic and so are — roughly speaking — their algebras of operators. This means that all physical systems would be equivalent, which

is obviously not the case (unless one has only one physical system, which then would have to be the microphysical world).

In every practical case one has to consider an isolated physical system which — though it has to be large enough so that it does not become trivial — must be small enough that one may command a view of its mathematical image.

The various microphysical systems in nature have quite distinct properties and must therefore have quite distinct mathematical images. Or, in other words, for a particular system some observables are more physical than others, and for a certain isolated physical system some observables are completely unphysical; e.g. if the system consists of the bound states of the hydrogen atom, then for this physical system one can never prepare a state in which the electron position is in a narrow domain of space, because a measurement of the observable position will destroy this physical system.

Thus for a particular physical system not all mathematical observables are physical observables. Further, for a particular physical system some observables appear to be more basic than others, and which observables are more basic depends upon the particular physical system.

Therefore it appears appropriate to make a distinction

between observables and operators.

An observable is a physical quantity and the physical definition of an observable consists of either a prescription for measuring the quantity itself or a definite expression for it in terms of other measurable quantities.

If not all operators are observables then also not all vectors can be physical states, because the preparation of a physical state constitutes the measurement of one or more observables, and a pure state that results from this measurement is an eigenstate of the observables. Thus eigenstates of operators which are not observables cannot be prepared.

Further— and this is connected with 2.—there are vectors which are connected with operators that are observables but which are not physically realizable. These are the vectors that lie outside the domain of definition of an observable that is represented by an unbounded operator, e.g. the vector which would correspond to the state of infinite energy.

With regard to 2.): The Hilbert space is infinite dimensional and it is infinite dimensional in a very particular sense, namely it is complete with respect to a very particular topology, i.e. with respect to a very particular meaning of convergence of infinite sequences. Since an infinite number of states can never be prepared, physical measurements

cannot tell us anything about infinite sequences, but can at most give us information about arbitrarily large but finite sequences. Therefore physics cannot give us sufficient information to tell how to take the limit to infinity, i.e., how to choose the topology.

On the other hand, we are also not permitted to choose a particular finite dimension N for the space of physical states, because after one has made N experiments one can always perform an $(N+1)^{th}$ experiment. Therefore the dimension of the space must be larger than any given finite number.

I believe that the choice of the topology will always be a mathematical generalization, but then one should choose the topology such that it is most convenient.

At the time when quantum mechanics was developed the Hilbert space was the only linear topological space that had structures which were required by quantum mechanics. That is, it is linear to fulfill the superposition principle, and it could accommodate infinite dimensional matrices and differential operators. Therefore it was natural that von Neumann chose the Hilbert space when he wanted to give a mathematically rigorous formulation of quantum mechanics. With this choice, though, he could not accommodate all the

features of Dirac's formulation[4] of quantum mechanics, which von Neumann says is "scarcely to be surpassed in brevity and elegance" but "in no way satisfies the requirements of mathematical rigour".

Since that time, stimulated by the Dirac formalism, a new branch of mathematics, the theory of distributions, has been developed. The abstraction of this theory provided the mathematical tool for a mathematically rigorous formulation of quantum mechanics, which not only overcomes the above mentioned shortcomings of von Neumann's axioms but also has all the niceties of the Dirac formalism.

This mathematical tool is the Rigged Hilbert Space, which was introduced around 1960 by Gelfand[1] and collaborators and Maurin[2] in connection with the spectral theory of self-adjoint operators. Its use for the mathematical formulation of Quantum Mechanics was suggested around 1965 by J. E. Roberts[5] and A. Bohm[6]; the suggestion to choose the topology such that the operators are continuous was made (for the canonical commutation relation) by Kristensen, Meljbo and Thue Poulsen[3].

We formulate the general framework by replacing the von Neumann axiom which states the one-to-one correspondence between observables and self-adjoint operators in the Hilbert space by the following basic assumption[7].

A physical observable is represented by a linear operator in a linear space (space of states).

The mathematical image of a physical system is an algebra A of linear operators in a linear scalar product space. The linear space is equipped with the weakest nuclear topology that makes this algebra an algebra of continuous operators. That is, if X_i, $i = 1,2,..n$ are the generators of A, then the operator $\Delta = \sum_{i=1}^{n} X_i^2$ is essentially self-adjoint (e.s.a.).

We emphasize again: An observable is the physical quantity which is defined by the prescription for its measurement. The algebra is the mathematical structure which is defined by the algebraic relations between its generators which represent some basic physical quantities and by other mathematical conditions, like the space in which the elements of the algebra act as operators.

The connection between the quantities calculated in the mathematical image, such as the matrix elements $(\psi, A\psi)$ or $\mathrm{Tr}(AW)$ and the quantities measured in the physical world, such as the expectation value (average value of a measurement) $M(A)$, is given by $M(A) = (\psi,A\psi)$ or in general $M(A) = \mathrm{Tr}(AW)$. These connections are stated by further basic assumptions which we will not discuss in this paper.

Within the framework of the above stated basic assumption the actual physical question is how to find the algebra. In non-relativistic quantum mechanics one can usually obtain

this algebra from correspondence with the corresponding classical system. When no corresponding classical system exists one has to conjecture this algebra from the experimental data.

1. The Algebraic Structure of the Space of States

We shall treat in detail one of the simplest physical models, the one-dimensional harmonic oscillator, in the framework of the basic assumption stated in the preceeding section. We will give a formulation which easily generalizes to more complicated physical models, and shall remark for which algebras these generalizations are already known.[8]

The algebra[9] of operators for the one-dimensional harmonic oscillator is generated by the operators

H representing the observable energy

(1) P representing the observable momentum

Q representing the observable position

and the defining algebraic relations are

$$PQ - QP = \frac{\hbar}{i} \mathbf{1} \tag{1}$$

$$H = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2$$

$\hbar$, m, ω are constants, universal or characteristics of the system.

A is an algebra of linear operators in a linear space, Φ, with scalar product[10], $(\cdot,\cdot)$(but Φ is not a Hilbert space) Further: P,Q,H are symmetric operators, i.e.

(2) $(P\phi,\psi) = (\phi,P\psi)$ for all $\phi,\psi \in \Phi$.

(1) and (2) do not specify the mathematical structure

completely. There are many such linear spaces in which $\mathcal{A}$ is an algebra of operators (whether the requirement that its topology be nuclear and the weakest puts sufficient restrictions is unknown). Therefore we have to make one further assumption to specify the algebra of operators, i.e. the representation of (1) and (2). This requirement can be formulated in two equivalent ways:

(3a) H is essentially self adjoint[11] (e.s.a.)

or

(3b) *There exists at least one eigenvector of H.*

We will use this requirement in the form (3b), as it is this form in which it is always used by physicists.

Remark:

We remark that either of these requirements leads to a representation of (1) which integrates to a representation of the group generated by P,Q,1 (Weyl group);

3a) by a theorem of Nelson[12] and

3b) because it leads to the well known ladder representation and ladder representations are always integrable[13] (*in this particular case one can easily see that* H in the ladder representation will be e.s.a.).

The generalization of (3) is the requirement that $\Delta = \Sigma X_i^2$ be e.s.a. For the case that $\mathcal{A}$ is the enveloping algebra of a Lie group, this will always lead

to an integrable representstion (Nelson theorem)[12], i.e. a representation that is connected with a representation of a group (even though no symmetry may be involved). o

The procedure to find the ladder representation is well known and will be sketched only very briefly:

One defines

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, Q + \frac{i}{\sqrt{m\omega\hbar}}\, P\right)$$

(4)

$$a^{+} = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, Q - \frac{i}{\sqrt{m\omega\hbar}}\, P\right)$$

(5) $N = a^{+}a = \frac{1}{\omega\hbar} H - \frac{1}{2} \mathbf{1}$

These operators clearly fulfill

$(\phi, a\psi) = (a^{+}\phi, \psi)$ for every $\phi, \psi \in \Phi$

and

$(\phi, N\phi) = (N\phi, \psi)$ for every $\phi, \psi \in \Phi$

as a consequence of (2).

As a consequence of (1) a and a^{+} fulfill

(6) $a\, a^{+} - a^{+} a = \mathbf{1}$

(7a) As a consequence of (3a) N is e.s.a.

(7b) or as consequence of (3b) there exist ϕ_λ such that

(7b) $N\phi_\lambda = \lambda\phi_\lambda$

From the c.r. (6) then follows

$$N(a\phi_\lambda) = a^+ a a \phi_\lambda = (a a^+ - 1) a \phi_\lambda = a(a^+ a - 1) \phi_\lambda =$$

(8) $$= a(N-1)\phi_\lambda = a(\lambda-1)\phi_\lambda = (\lambda-1)(a\phi_\lambda)$$

i.e. $a\phi_\lambda$ is an eigenvector with eigenvalue $(\lambda-1)$

or $a\phi_\lambda = 0$

Further from c.r. (6) follow

$$||a^+\phi_\lambda||^2 = (\phi_\lambda, aa^+\phi_\lambda) = (\phi_\lambda, a^+a\,\phi_\lambda) + (\phi_\lambda, 1\phi_\lambda)$$

$$= ||a\phi_\lambda||^2 + ||\phi_\lambda||^2 \neq 0 \text{ as } \phi_\lambda \neq 0$$

i.e.

(9) $a^+\phi_\lambda \neq 0$ always.

Further follows from (6)

(10) $N(a^+\phi_\lambda) = (\lambda+1)\, a^+\phi$

i.e. $a^+\phi_\lambda$ is always an eigenvector of N with eigenvalue $(\lambda+1)$. Starting with ϕ_λ, which was <u>assumed</u> to exist, one can obtain eigenvectors

$$\phi_{\lambda-m} = a^m\,\phi_\lambda \qquad m = 0,1,2,\ldots$$

with eigenvalue $(\lambda-m)$ of N

$$N\,\phi_{\lambda-m} = (\lambda-m)\,\phi_{\lambda-m}$$

After a finite number of steps

$$a^m \phi_\lambda = 0$$

Proof:

$$(\phi_{\lambda-m}, N\, \phi_{\lambda-m}) = (\lambda-m)(\phi_{\lambda-m}, \phi_{\lambda-m}) = (\phi_{\lambda-m}, a^+ a\ \phi_{\lambda-m}) = ||a\ \phi_{\lambda-m}||^2$$

therefore

(11) $$(\lambda-m) = \frac{||a\phi_{\lambda-m}||^2}{||\phi_{\lambda-m}||^2} \geq 0$$

i.e. there exists a $\phi_{\lambda-m}$ such that

$$a\phi_{\lambda-m} = 0$$

We call ϕ_o the normalized vector for which $a\phi_o = 0$

i.e. $\phi_o = \dfrac{\phi_o}{||\phi_o||}$;

(12) $$N\Phi_o = 0$$

Then one defines the states

$$\phi_o$$

$$\phi_1 = \frac{1}{\sqrt{1!}} a^+ \phi_o$$

(13) $$\phi_2 = \frac{1}{\sqrt{2!}} (a^+)^2 \phi_o$$

$$|$$

$$|$$

$$|$$

$$\phi_n = \frac{1}{\sqrt{n!}} (a^+)^n \phi_o$$

which have the property

$$\text{(14)} \quad {}^{1)}\|\phi_n\| = 1 \qquad {}^{2)}N\phi_n = n\phi_n \qquad {}^{3)}(\phi_n,\phi_m) = \delta_{nm}$$

${}^{4)}$For every ϕ_n there exists a $\phi_{n+1} \neq 0$.

The linear space spanned by $\phi_o,\phi_1\ldots\phi_n\ldots$ we call Ψ, i.e.

Ψ is the set of all vectors

$$\Psi = \sum_{n=0,1\ldots}^{m} \alpha^{(n)} \phi_n$$

where $\alpha^{(n)} \in \mathbb{C}$ and m is a natural number which is arbitrarily large but finite.

We call

R_i the space spanned by ϕ_i, i.e.

$R_i = \{\alpha\phi_i | \alpha\in\mathbb{C}\}$, ($R_i$ are the energy eigenspaces).

Using R_i instead of ϕ_i we have a formulation which immediately applies when R_i is not one dimensional. Then Ψ is the algebraic direct sum of the R_i

$$\Psi = \sum_{alg.} R_i$$

In other words Ψ is the set of all sequences

$\phi=(\phi_o,\phi_1,\phi_2,\ldots\phi_m,000\ldots)$ with $\phi_i \in R_i$

where the algebraic operations of the linear space are defined by

$\phi + \psi = (\phi_o + \psi_o, \phi_1 + \psi_1, \ldots)$

$\alpha\phi = (\alpha\phi_o, \alpha\phi_1 \ldots)$

The scalar product is

$$(\phi,\psi) = \sum_{i=o}^{m} (\phi_i,\psi_i)$$

And the norm is

$$||\phi||^2 = \sum_{i=0}^{m} (\phi_i,\phi_i)$$

As the sum goes only up to an arbitrarily large but finite number m the question of convergence of the sum of numbers (ϕ_i,ϕ_i) does not arise.

Ψ is a linear space without any topological structure. We now equip Ψ with a topology, i.e. we construct a linear topological space.

II. The Topological Structure of the Space of States

A set Φ is a linear topological space (in which the first axiom of countability is fulfilled) iff

I) Φ is a linear space

II) For a sequence of elements of Φ the notion of a limit element is defined (the limit is unique and a subsequence of a convergent sequence converges to the same limit point)

III) The algebraic operations of the linear space are continuous

i.e. 1)a) If $\Phi \ni \phi_n \to \phi \in \Phi$ then also $\alpha\phi_n \to \alpha\phi$ for $\alpha \in \mathbb{C}$

b) If $\mathbb{C} \ni \alpha_n \to \alpha \in \mathbb{C}$ then also $\alpha_n \phi \to \alpha\phi$ for every $\phi \in \Phi$

2) If $\phi_n \to \phi$ and $\psi_n \to \psi$ then $\phi_n + \psi_n \to \phi + \psi$ for

$\phi_n, \psi_n, \phi, \psi \in \Phi$; $n = 1,2,\ldots.$

This is not the most general definition of a linear topological space but it is sufficient for our present purpose (we will later on also use spaces in which the first axiom of countability is not fulfilled)

In a given linear space the convergence can be defined in various ways leading to various linear topological spaces. All the topological notions like continuity, denseness, boundedness, closure, completeness.. depend then on the definition of convergence.

We shall introduce into Ψ three different topologies.

As a first example we introduce into Ψ the well known Hilbert space topology τ_H (one can check that this really fulfills the conditions 1a) 1b) and 2) of the definition.)

Def:[14)]

(1) $\phi_\gamma \to \phi$ for $\gamma \to \infty \quad \leftrightarrow \quad ||\phi_\gamma - \phi|| \to 0 \quad$ for $\gamma \to \infty$.

To understand the relation between Ψ and the Hilbert space H let us introduce the following notion:

Def: A sequence $\{x_n\}$ of elements in a space with a norm (Normed Space) is called a Cauchy sequence with respect to τ_H if for every $\varepsilon > 0$ there exists

(2) an N such that $||x_n - x_m|| < \varepsilon$ for every $m,n > N$.

Def: A linear topological space R is called complete if every Cauchy sequence $\{x_n\}$ has a limit element x which is an element of the space R.

If a space is not complete it can be completed by adjoining all limit elements of Cauchy sequences to it.

It is easy to see that Ψ is not τ_H-complete: Let us consider the infinite sequence

(3) $h = (h_1, h_2, h_3 \ldots h_i \ldots)$ with $h_i \in R_i$

$h_i \neq 0$ for all i

and $\sum\limits_{i=0}^{\infty} ||h_i||^2 < \infty$

From this follows

$$||h_n|| \to 0 \quad \text{for } n \to \infty \qquad \text{and}$$

$$\sum_{i=n+1}^{\infty} ||h_i||^2 \to 0 \qquad \text{for } n \to \infty$$

h is <u>not</u> an element of Ψ, because it is an infinite sequence.

Let us consider the sequence

$$S_1, S_2, S_3 \dots \quad \text{with } S_i \in \Psi$$

where

a) $S_n = (h_1, h_2, \dots h_n, 0, 0 \dots)$

As $S_n - S_m = (0,0, \dots 0, h_{n+1}, h_{n+2} \dots h_m, 00 \dots)$

we have

$$||S_n - S_m|| = ||h_{n+1} + h_{n+2} + h_m|| \to 0 \text{ for } n \to \infty$$

i.e. $\{S_n\}$ is a Cauchy sequence.

The τ_H-limit element of S_n is h:

$$S_n \xrightarrow{\tau_H} h$$

because $||S_n - h||^2 = \sum_{i=n+1}^{\infty} ||h_i||_i^2 \to 0$ for $n \to \infty$

But $h \notin \Psi$.

If we adjoin to Ψ all the limit elements of Cauchy sequences,

i.e. take in addition to the elements

$\phi = (\phi_1, \phi_2, \ldots, \phi_n\ 0\ 0\ 0 \ldots)$, n arbitrarily large, the elements h of the kind defined in (3), then we obtain a complete normed space in which the norm is given by the scalar product; this is the Hilbert space H. Thus, the Hilbert space H is the completion[15] of Ψ with respect to the topology defined by (3).

Def. A subspace S of a complete topological space X is called dense if

$$S \cup \{\text{all limit points of } S\} = X.$$

Thus Ψ is a τ_H-dense subspace of H.

The space H is the space of all $h = (h_o, h_1 \ldots)$, $h_i \in R_i$ which fulfill (3); this space is written as

$$H = \sum_{i=o}^{\infty} \oplus R_i$$

and is called the Hilbertian direct sum or orthogonal direct sum of the R_i.

Thus the Hilbertian direct sum is the completion of the algebraic direct sum with respect to τ_H.

We now introduce into Ψ a new topology which we call τ_Φ. τ_Φ is defined in the following way:

We take the original scalar product $(\cdot\, , \cdot)$ and define

the following quantities

$$(6)\quad (\phi,\psi)_p = (\phi,(N+1)^p\psi) \qquad p = 0,1,2,\ \ldots$$

and $||\phi||_p = \sqrt{(\phi,\psi)_p}$.

Remark: Before proceeding we note some properties of the operator N: N is essentially self-adjoint. This is easily proved using one of the criteria of essentially self-adjointness:

Lemma: An operator A is e.s.a. if $(A+1)^{-1}$ is continuous and has a dense domain in H.

The spectrum of $(N+1)^{-1}$ is $\frac{1}{n+1}$, consequently it is continuous. Its domain is Ψ, which is dense in H. Consequently N is e.s.a.. As a consequence N+1 is e.s.a. Further $(N+1)^p$ for every p is e.s.a.[16)]

From the spectrum of $(N+1)^p$ one sees immediately that $(N+1)^p$ is positive definite. From this it is easy to see that $(\phi,\psi)_p$ for every p fulfills the condition of a scalar product and

$$(6a)\quad ||\phi||_0 \le ||\phi||_1 \le ||\phi||_2 \le \cdots$$

Remark: Further, these norms are compatible, i.e.. if a sequence converges with respect to one norm and is a Cauchy sequence with respect to another, then it also converges with respect to this other norm. o

A space in which a countable number of norms (scalar-

products) are defined is called a countably normed (countably scalar product) space.

We now define τ_Φ by:

Def:

(7) $\phi_\gamma \xrightarrow{\tau_\Phi} \phi \leftrightarrow ||\phi_\gamma - \phi||_p \to 0$ for every p.

From this definition one immediately sees

(8) From $\phi_\gamma \xrightarrow{\tau_\Phi} 0 \to$ follows $\phi_\gamma \xrightarrow{\tau_H} 0$ but not vice versa.

Therefore τ_Φ is called stronger (finer) than τ_H, and τ_H is called weaker (coarser) than τ_Φ.

Def: A sequence $\{x_\gamma\}$ is called a τ_Φ-Cauchy sequence if for every p and for every $\epsilon > 0$ there exists an $N = N(\epsilon, p)$ such that

(9) $||x_\gamma - x_\mu||_p < \epsilon$ for every $\mu, \gamma > N$.

There are more τ_H-Cauchy sequences than τ_Φ-Cauchy sequences because for τ_H - Cauchy sequences (9) needs to be fulfilled only for $p = 0$.

We complete[15] Ψ with respect to τ_Φ, i.e. adjoin to Ψ the limits points of all τ_Φ-Cauchy sequences. The linear topological space that we obtain by completing Ψ with respect to τ_Φ we call Φ.

Then

$\Psi \subset \Phi$, Ψ is τ_Φ-dense in Φ.

Φ is called a countably Hilbert space. As there are more τ_H-Cauchy sequences than τ_Φ-Cauchy sequences it must be that

(10) $$\Psi \subset \Phi \subset H.$$

Thus Ψ is τ_Φ-dense in Φ

Ψ is τ_H-dense in H

and Φ is τ_H- dense in H, as already Ψ is τ_H- dense in H .

To get a feeling for Φ let us see which of the infinite sequences (3) that are elements of H are also elements of Φ. In order that $h = (h_o,h_1,..) \in H$ be a limit point of a τ_Φ-Cauchy sequence $\{S_n\}$ with $S_n = (h_1\ h_2\ \dots\ h_n\ 000) \in \Psi$

i.e. in order that:

$$S_n \xrightarrow{\tau_\Phi} h$$

one must have

$$||S_n - h||_p \to 0 \quad \text{for } \underline{\text{every } p}, \text{ i.e.}$$

(11) $$((S_n-h),(N+1)^p(S_n-h)) \to 0 \qquad \textit{for every } p.$$

i.e., one must have

$$\sum_{i=n+1}^{\infty} (h_i,(N+1)^p\, h_i) = \sum_{i=n+1}^{\infty} (i+1)^p||h_i||^2 \to 0 \quad \text{for every } p.$$

Therefore, only those $h \in H$ are τ_Φ-limit points of τ_Φ-Cauchy sequences of elements of Ψ —that means elements of Φ— for which

$$\sum_{i=o}^{\infty} (i+1)^p||h_i||^2 < \infty .$$

Φ is called the τ_Φ-direct sum of R_n and is written

$$12)\quad \Phi = \sum_{n=0}^{\infty} \tilde{\oplus}\, R_n$$

To see what the use of the various spaces might mean for physics we recall that R_n are the energy eigenspaces with energy value

$$(13)\quad E_n = \hbar\omega(n + \tfrac{1}{2})$$

Thus Ψ describes states with arbitrarily high but finite energy. H contains the "infinite energy" state, and Φ contains states which have an "infinitely small admixture of infinite energy states".

Clearly physics can only tell us something about Ψ. In fact for most real physical systems, e.g. diatomic molecules, whose idealization is the harmonic oscillator, only the very lowest energy levels are relevant; for higher energy the diatomic molecule is no longer a harmonic oscillator and finally not even an oscillator. H and Φ are mathematical idealizations, though Φ appears "closer" to reality.

The question there is, why do we want these mathematical idealizations and why do we have a preference for one over the other.

The advantage that Φ has over H is mathematical convenience,

which can be vaguely summarized by saying Φ admits the Dirac formalism. Two aspects of this formalism are: 1) All algebraic operations with the operators are allowed and no questions with respect to the domain of definition arise. 2) For every e.s.a. operator there exists a complete system of generalized eigenvectors. 1) follows from the fact that all elements of the algebra A are continuous operators with respect to τ_Φ and therefore uniquely defined on the whole space Φ.[17)]

To show this it is sufficient to show that P and Q or a and a^+ are continuous because of the Theorem: The product and sum of continuous operators are continuous operators.

Thus in general an algebra is an algebra of continuous operators iff the generators are continuous operators.

We recall that (in spaces in which the topology can be defined by the convergence of sequences (i.e. spaces with the first axiom of countability)) an operator A is continuous if for all sequences $\{\phi_\gamma\}$ with $\phi_\gamma \to 0$ it follows that $A\phi_\gamma \to 0$. To prove that a (and a^+) is τ_Φ-continuous let us consider a and a^+ on Ψ.

We use the <u>lemma</u> (proved in Appendix I): For every $\phi \in \Psi$

$$(14)\quad (\phi, a(N+1)^p a^+\phi) \leq k(\phi, (N+1)^{p+1}\phi)$$

where k is some constant, $k < \infty$.

Proof: Let $\phi_\gamma \xrightarrow{T_\phi} 0$ as $\gamma \to \infty \leftrightarrow ||\phi_\gamma||_p \to 0$ for every p

(15) $\leftrightarrow (\phi_\gamma,(N+1)^p \phi_\gamma) \to 0$ for every p.

To show that

$$a^+ \phi_\gamma \xrightarrow{T_\phi} 0 \text{ we have to show that}$$

(16) $||a^+ \phi_\gamma||_q \to 0$ for every q. i.e. that

$$(a^+\phi_\gamma,(N+1)^q a^+\phi_\gamma) = (\phi_\gamma, a(N+1)^q a^+\phi_\gamma) \to 0 \text{ for every } q.$$

By (14):

$$(\phi_\gamma, a(N+1)^q a^+\phi_\gamma) \leq k(\phi_\gamma,(N+1)^{q+1}\phi_\gamma)$$

but by (15) the r.h.s. of this inequality $\to 0$, consequently also the l.h.s. which proves (16).

We remark that the convergence of $||a^+ \phi_\gamma||_q \to 0$ for $\gamma \to \infty$ for a fixed q follows from the convergence of $||\phi_\gamma||_{q+1} \to 0$. Therefore in the case of a finite number of norms a^+ will not be a continuous operator. As the topology in the case of a finite number of norms is equivalent to the topology given by the highest norm (i.e. sequences that converge with respect to one converge also with respect to the other and vice versa) this fact implies that a^+ cannot

be a continuous operator in the Hilbert space topology, which is well known. The proof of the continuity of a is analogous. Therewith we have shown that a and a^+ are continuous operators on the dense subspace $\Psi \subset \Phi$ and can therefore be uniquely extended to operators in the whole space Φ.[18] We denote the operators a, a^+, P, Q.. extended to the whole space Φ again by a, a^+, P, Q... a and a^+ and therewith the whole algebra are defined on the whole space Φ and domain questions do not arise in the algebraic operations.

The τ_Φ-continuous operators P, Q, H considered as operators in $\mathcal{H}$ are not closed operators[19] (they are τ_Φ-continuous and consequently τ_Φ-closed but not $\tau_{\mathcal{H}}$-closed), because if $f_\gamma \in \Phi$ and $f_\gamma \xrightarrow{\tau_{\mathcal{H}}} f$ but $f \notin \Phi$, then $Af_\gamma \in \Phi$ for every γ but A is not defined on f. If $Af_\gamma \xrightarrow{\tau_{\mathcal{H}}} g$ then we define $\bar{A}f$ by

$$\bar{A}f = g.$$

We can do this for all $f \in \mathcal{H}$ which are $\tau_{\mathcal{H}}$-limit points of some sequences $f_\gamma \in \Phi$ and for which Af_γ $\tau_{\mathcal{H}}$-converges. The operator $\bar{A}$ defined in this way, i.e. whose domain $D(\bar{A})$ is the set of all these $f \in \mathcal{H}$, is the closure of A in the completion of Φ to $\mathcal{H}$.

Thus in correspondence to the relation (10) between the spaces:

$$\Phi \subset H$$

we have the relation

$$A \subset \bar{A} \tag{17}$$

between the operators.

(however in general $D(\bar{A}) \neq H$).

So far we know of the operators P,Q,H(or N) that they are τ_Φ-continuous symmetric operators on Φ.

Therefore $\quad P^+ \supset \bar{P} \quad$ (i.e. $D(P^+) \supset D(\bar{P})$)

(18) $\quad$ and $\quad Q^+ \supset \bar{Q}$.

H(and N) is e.s.a.*) and therefore

(19) $\qquad H^+ = \bar{H} \qquad (N^+ = \bar{N})$

However as a consequence of (19) it follows that also P,Q, are e.s.a., i.e.:

(20) $\quad P^+ = \bar{P} \qquad\qquad D(P^+) = D(\bar{P})$

$\qquad Q^+ = \bar{Q}$

Remark:

H+I is, except for some constant factors, the Nelson operator, which is because of (19) e.s.a.. Therefore —as mentioned before— P,Q,I are the group generators of a group of transformations (Weyl Group).

As a consequence of the fact that P,Q are elements of the Lie algebra of this group it follows then that P and Q are also e.s.a., by a theorem of Nelson and Stinespring.*)[20].

We now turn to the 2nd aspect of the Dirac formalism, the existence of a complete set of generalized eigenvectors, which follows from the fact that for the operators in Φ the nuclear spectral theorem applies. The explanation of this requires some further mathematical preparations.

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*) Either as a consequence of assumption (3b) as proven in Remark on p. 12 or by assumption (3a).

Def: A bounded self-adjoint operator B is Hilbert-Schmidt iff $B = \Sigma\lambda_k P_k$ where the projections P_k project on finite dimensional spaces H_k and $\Sigma(|\lambda_k| \dim H_k)^2 < \infty$

Instead of giving the original definition of a nuclear space[21] we shall use a theorem of Roberts[22] which gives a necessary and sufficient condition for Φ to be nuclear.

Def: Φ is a nuclear space iff there exists an e.s.a. τ_Φ- continuous operator $A \in A$, whose inverse is Hilbert Schmidt.

It is now very easy to see that our Φ is nuclear, because N is e.s.a., the spectrum of N^{-1} is $\frac{1}{n}$, n = 1,2,3,... and R_n is one dimensional, thus $\Sigma(\frac{1}{n})^2 < \infty$. Thus N is the operator that fulfills the condition of the above definition.

Summarizing, the space Φ constructed above is a linear topological nuclear space in which all elements of the algebra are continuous operators.

Remark concerning generalizations:

The construction of a nuclear space described above is immediately generalized to a more general algebra. The analogue of the Lemma (14)

14') $(\phi, \chi\,(\Delta+1)^p\, X\,\phi) \leq k(\phi(\Delta+1)^{p+1}\phi)$

where X is one of the generators X_i and $\Delta = \Sigma X_i^2$ (Nelson operator) holds for all enveloping algebras (Lemma by Nelson). Therewith the continuity of the algebra in a linear topological space in which the topology is defined by the countable number of scalar

products

$$(\phi,\psi)_p = (\phi,(\Delta+1)^p\psi)$$

follows immediately. Further if Δ is e.s.a. then all the symmetric[11] generators are also e.s.a. (Theorems by Nelson and Nelson and Stinespring).

(14') is much stronger than required for the proof of the continuity of the generators. The continuity of the generators (and therewith the whole algebra) can already be proved if instead of $p + 1$ on the r.h.s. of (14') one has $p+n$, where n is any finite integer. Therefore it appears that the continuity of the generators can already be proved for any finitely generated associative algebra.

The nuclearity is a much harder property to establish. It has been proven for the cases that the algebra is the enveloping algebra $E(G)$ of the following groups G:

G nilpotent (because then $E(G)$ is isomorphic to the enveloping algebra generated by P_α, Q_α, $\alpha = 1,2,\ldots m$ with $[P_\alpha,Q_\beta] = \frac{1}{i}\,\delta_{\alpha\beta}I$ for some m (Theorem by Kirillov)[23] and we have just an m dimensional generalization of the above described case)

G semi-simple (A. Böhm)[24]

$G = A \rtimes K$ semidirect product, with A abelian and K compact. (B. Nagel)[25].

G Poincare group for some of the representations (B. Nagel)[25].

III Conjugate Space of Φ

An antilinear functional F on a linear space ψ is a function, the values $F(\phi) = \langle\phi|F\rangle$ of which are complex numbers, which satisfies:

(1) $F(\alpha\phi+\beta\psi) = \bar{\alpha}F(\phi) + \bar{\beta}F(\psi) \quad \phi,\psi \in \Psi;\alpha,\beta \in \mathbb{C}$

or in the other notation

(1) $$\langle\alpha\phi+\beta\psi|F\rangle = \bar{\alpha}\langle\phi|F\rangle + \bar{\beta}\langle\psi|F\rangle$$

The functional F is called τ-continuous iff from

(2) $$\phi_\gamma \xrightarrow{\tau} \phi, \gamma \to \infty \text{ follows } F(\phi_\gamma) \xrightarrow{\mathbb{C}} F(\phi)$$

where $\xrightarrow{\mathbb{C}}$ means convergence of complex numbers (for every $\varepsilon > 0$ there exists an $(q,\delta)(\varepsilon)$ such that $|F(\phi)| < \varepsilon$ for all ϕ with $||\phi||_q < \delta$).

The set of antilinear functionals on a space Ψ may be added and multiplied by numbers according to

(3) $$\langle\phi|\alpha F_1+\beta F_2\rangle = \alpha\langle\phi|F_1\rangle+\beta\langle\phi|F_2\rangle$$
$$(\alpha F_1+\beta F_2)(\phi) = \alpha F_1(\phi)+\beta F_2(\phi)$$

The functional

$\alpha F_1+\beta F_2$ defined by (3) is again a linear functional. Thus the set of antilinear functionals on a linear space Ψ forms a linear space. This space is called the algebraic dual or algebraic conjugate space and denoted Ψ^X.

Continuous antilinear functionals have not only to fulfill (1) but also (2), therefore the set of continuous antilinear functionals, which also forms a linear space, is smaller than the set Ψ^X and depends — like all topological notions — on the particular topology that has been chosen. We shall denote the set of all τ_Φ- continuous functionals by Φ^X and the set of all τ_H-continuous functionals by H^X. As every sequence $\{\phi_\gamma\}$ with $\phi_\gamma \xrightarrow{\tau_\Phi} 0$ has the property $\phi_\gamma \xrightarrow{\tau_H} 0$ i.e. there are more τ_H-convergent sequences than τ_Φ-convergent sequences and $F(\phi_\nu) \to F(0)) = 0$ must be fulfilled for all τ_H-convergent sequences if $F = F^{(H)} \varepsilon H^X$, and for all τ_Φ-convergent sequences if $F = F^{(\Phi)} \varepsilon \Phi^X$ the set of $F^{(H)} \varepsilon H^X$ will be smaller than the set of $F^{(\Phi)} \varepsilon \Phi^X$ (because there are more stringent conditions on elements of H^X). Therefore: H^X is the smallest set and Ψ^X is the largest set of the set: H^X, Φ^X, Ψ^X.

In a scalar product space each vector $f \varepsilon \Psi$ defines an antilinear functional by

$$(5) \langle\phi|F\rangle = F(\phi) \overset{\text{def}}{=\!=} (\phi, f)$$

It is easy to see that F defined by the vector f as given in (5) fulfills the condition (1) if (ϕ, f) fulfills the conditions for a scalar product.

Further if $\phi_\gamma \to \phi$ then for every $f \in \Psi$ (even for every $f \in H$) $(\phi_\gamma, f) \to (\phi, f)$ (every strongly convergent sequence converges also weakly). Thus the functional defined by (5) fulfills $F(\phi_\gamma) \to F(\phi)$, i.e. it is a continuous antilinear functional. Thus the scalar product defines a continuous antilinear functional.

In the Hilbert space the converse is also true, i.e. any antilinear τ_H-continuous functional can be written as a scalar product according to the Frechet-Riesz theorem:

For every antilinear τ_H-continuous functional $F^{(H)}$ there exists a unique vector $f \in H$ such that

(6) $\langle\phi|F^{(H)}\rangle = F^{(H)}(\phi) = (\phi, f)$ for all $\phi \in H$

Therefore we can identify the Hilbert space H and its conjugate H^X by equating $F \in H^X$ with the $f \in H$ given by (6).

Then with (II,9) we have the situation:

(7) $\Psi \subset \Phi \subset H = H^X$

For τ_H-continuous functionals $F^{(H)}$ the symbols $\langle|\rangle$ and $(,)$ are the same, after the identification $F^{(H)} = f$:

(9) $\langle\phi|f\rangle = (\phi, f)$

However for the larger class of τ_Φ-continuous functionals F the symbol $\langle\phi|F\rangle$ is defined, whereas (ϕ, F) is not unless $F \in \Phi^X$ is already $\in H^X$ in which case we identify:

(10) $\langle\phi|F\rangle = \langle\phi|F^{(H)}\rangle = (\phi, F)$

In Φ^X one can introduce various topologies and therewith various meanings of convergence of countable sequences. An example is the weak convergence (in analogy to the weak convergence in H):

Def: A sequence of functionals $\{F_\gamma\} \subset \Phi^X$ converges (weakly) to a functional F iff

$$(11)\quad \langle\phi|F_\gamma\rangle \to \langle\phi|F\rangle, \qquad \gamma \to \infty \quad \text{for every } \phi \in \Phi.$$

One can show that Φ^X is already complete with respect to the topology τ^X of this convergence.

Remark: Φ^X is a space in which the topology cannot be completely described by the description of the passage to the limit of countable sequences (Φ^X does not satisfy the first axiom of countability and is, therefore, a more general topological space than those defined by our definition in II). Therefore we cannot attempt to prove the following statements.[26)]

As Φ^X is a linear topological space we can consider the τ^X-continuous antilinear functionals $\tilde{\phi}$ on Φ^X, i.e. the set Φ^{XX} of all those functions on Φ^X which satisfy

$$(1^X)\quad \tilde{\phi}(\alpha F_1+\beta F_2) = \bar{\alpha}\,\tilde{\phi}(F_1) + \bar{\beta}\,\tilde{\phi}(F_2), F_1, F_2 \in \Phi^X;\ \alpha,\beta \in \mathbb{C}$$

or in other notation

$$\langle\alpha F_1+\beta F_2|\tilde{\phi}\rangle = \bar{\alpha}\langle F_1|\tilde{\phi}\rangle+\bar{\beta}\langle F_2|\tilde{\phi}\rangle$$

and

(2^{x}) $\tilde{\phi}(F_\gamma) \to \tilde{\phi}(F)$ for every $F_\gamma \xrightarrow{\tau^{x}} F$

Φ^{xx} is a linear topological space if addition and multiplication are defined by:

$$\langle F|\alpha\tilde{\phi}_1+\beta\tilde{\phi}_2\rangle = \alpha\langle F|\tilde{\phi}_1\rangle + \beta\langle F|\tilde{\phi}_2\rangle$$

and (weak) convergence by:

(11^{x}) $\tilde{\Phi}_\gamma \xrightarrow{\tau^{xx}} \tilde{\phi} \leftrightarrow \langle F|\tilde{\phi}_\gamma\rangle \to \langle F|\tilde{\phi}\rangle$ for every $F \in \Phi^{x}$

One can prove[27)] that there corresponds an antilinear continuous functional $\tilde{\phi}$ on Φ^{x} to each element $\phi \in \Phi$ defined by the equation

$$(12) \quad \tilde{\phi}(F) = \overline{F(\phi)}$$

or

$$(12) \quad \langle F|\tilde{\phi}\rangle = \overline{\langle\phi|F\rangle}$$

and that the convergence defined by (11^{x}) agrees with the convergence with respect to τ_Φ. Thus, with the identification $\tilde{\phi} = \phi$ given by (12),

$$\Phi^{xx} = \Phi$$

(Φ is reflexive).

The Hilbert space H is certainly reflexive (because already $H^{x} = H$) and as the functionals in H are given by the scalar product the relation corresponding to (12) is the property

of the scalar product:

(13) $(f,h) = \overline{(h,f)}$

For all τ_Φ-continuous functionals F on Φ which are also τ_H -continuous functionals f on Φ we identify f with F or $\langle F| = (f|$. And as there are more τ_Φ-continuous functionals we have the situation

(14) $$H^X \subset \Phi^X$$

which with (7) gives

(15) $$\Phi \subset H \subset \Phi^X$$

This triplet is called Rigged Hilbert space or Gelfand Triplet.

For every continuous linear operator A in Φ one can define the adjoint operator A^X in Φ^X by:

(16) $$\langle\phi|A^X|F\rangle = F(A\phi) = \langle A\phi|F\rangle = \overline{\langle F|A\phi\rangle}$$

If A is a continuous operator in Φ and F is a continuous antilinear functional then A^X is a continuous operator in Φ^X i.e., it has in particular the property:

(17) $$A^X F_\gamma \to A^X F \quad \text{for all } F_\gamma \to F.$$

In particular the adjoint operators P^X, Q^X, H^X of the operators P,Q,H are τ^X-continuous operators in Φ^X.

Remark: We have defined a continuous operator only by the convergence of sequences. This is possible only in spaces with the first axiom of countability. Though we have not

given the general definition of continuous and bounded operators we shall remark on the connection between them as these two notions are used interchangeably. In spaces with the first axiom of countability every continuous operator is bounded and every bounded operator is continuous. The Hilbert space and the space Φ are such spaces. In general, e.g. in the space Φ^X, continuous operators are not bounded, and the condition (17) is only necessary for A^X to be a continuous map.o

For every τ_Φ-continuous symmetric operator A we have, in correspondence to the relation (8) between the spaces

15) $$\Phi \subset H \subset \Phi^X$$

18) the relation between the operators $A \subset \bar{A} \subseteq A^+ \subset A^X$,

and $$A \subset \bar{A} = A^+ \subset A^X$$

if A is e.s.a.

IV Generalized Eigenvectors and Nuclear Spectral Theorem

Before we define generalized eigenvectors of an operator A in the space Φ let us recall the situation in the Hilbert space $\mathcal{H}$.

A vector $h \in \mathcal{H}$ is called an eigenvector of an operator A on the Hilbert space iff

$$Ah = ah$$

where a is a number. In general, it is not possible to find for every operator on $\mathcal{H}$ an eigenvector. In fact it is well known that the operators $\overline{P}$ and $\overline{Q}$ do not have any eigenvector in $\mathcal{H}$. Yet physicists, following Dirac, always work with "eigenvectors" of P and Q

$$P|p\rangle = p|p\rangle$$

and use the assumption that these eigenvectors form a "complete" system in the sense that

$$\int |p\rangle\langle p| = 1$$

We will now give a definition and a theorem that provides a mathematical justification of this procedure.

Def: Let A be an operator in Φ. A generalized eigenvector of the operator A corresponding to the generalized eigenvalue λ is an antilinear functional, $F \in \Phi^{X}$, such that

(1) $F(A\phi) = \langle A\phi|F\rangle = \langle\phi|A^{X}|F\rangle = \overline{\lambda}\langle\phi|F\rangle$

holds for every $\phi \in \Phi$, which may be written in the form:

(1') $A^{X}|F\rangle = \overline{\lambda}|F\rangle$

We shall use this definition only for the case that A is e.s.a. (often one defines the generalized eigenvector by $F(A^+\phi) = \lambda F(\phi)$ where A^+ is the adjoint operator of A).

Let us assume A has generalized eigenvectors in the Hilbert space i.e. F in (1) is an element of $H^\times$. Then (1) reads

2) $\langle A\phi | F\rangle = (A\phi, f) = (\phi, A^+ f) = \bar{\lambda}(\phi, f)$

for every $\phi \in \Phi$ and consequently (since Φ is τ_H-dense in H)

3) $A^+ f = \bar{\lambda} f$ and if A is e.s.a. $A^+ f = \bar{A} f = \bar{\lambda} f = \lambda f$

Thus a generalized eigenvector which is an element of the Hilbert space is an ordinary eigenvector corresponding to the same eigenvalue (for the case that A is not e.s.a. it is an eigenvector of A^+).

Before we formulate the nuclear spectral theorem let us start with a few remarks.

Every self-adjoint operator A in a finite dimensional Hilbert space has a complete system of eigenvectors. I.e. there exists an orthonormal set of eigenvectors $h_{(\lambda_i)} = |\lambda_i) \in H$

4) $A\, h_{(\lambda_i)} = \lambda_i h_{(\lambda_i)}$ $\qquad (h_{(\lambda_i)}, h_{(\lambda_{i'})}) = (\lambda_i | \lambda_{i'}) = \delta_{i'i}$

such that for every $h \in H$

5) $h = \sum_{i=1}^{n} |\lambda_i)(\lambda_i | h)$ $\qquad n = \dim H$

The set of all eigenvalues $\Lambda = \{\lambda_1, \lambda_2, .. \lambda_n\}$ is called the spectrum of A.

For an infinite dimensional Hilbert space this statement is no more true, in fact we know that there are operators that have no eigenvector in H. (e.g.Q). The spectrum of an operator A in H is defined as the set Λ of all those λ for which $\bar{A}-\lambda 1$ has no inverse. The spectrum of a self-adjoint operator is a subset of the real axis[28]. If the self adjoint operator A in the infinite dimensional Hilbert space H has only a discrete spectrum,i.e. if $\Lambda = \{\lambda_1, \lambda_2, ..\}$ then the above statement carries over to the infinite dimensional case. To avoid complications which are inessential for the principal problems and inapplicable for the special problem of the one-dimensional harmonic oscillator we restrict ourselves to cyclic operators.

Def: A operator A is cyclic if there is a vector $f \in H$ such that $A^k f$, $k = 0,1,2,...$ span the entire Hilbert space. It is clear that the operator P and Q are cyclic because $Q^k \Phi_n = \sqrt{\frac{h}{2m\omega}}\,(a + a^+)^k\, \Phi_n$, $k = 0,1,2...$, any n span the entire H.

The generalization of the above statement to the case of a cyclic self adjoint operator with discrete spectrum is given by the following spectral theorem:

There exists an orthonormal set of eigenvectors

$$h_{\lambda i} = |\lambda_i) \in H$$

$$Ah_{\lambda_i} = \lambda_i h_{\lambda_i}, \quad \lambda_i \in \Lambda, \quad (h_{\lambda_i}, h_{\lambda_j}) = \delta_{\lambda_i \lambda_j}$$

such that for every $h \in H$

$$h = \sum_{i=1}^{\infty} h_{\lambda_i} (h_{\lambda_i}, h) = \sum_{i}^{\infty} |\lambda_i)(\lambda_i|h)$$

(8)

$$= \int_{\Lambda} d\mu(\lambda) \; |\lambda)(\lambda|h)$$

The measure μ is concentrated on the eigenvalue $\lambda_i \in \Lambda$ and one point sets have measure one, i.e. $\mu\{\lambda_i\} = 1$

The operator H is an example of such an operator. We remark that all eigenvectors (7) of A appear in the decomposition (8).

If A has continuous spectrum, then there are no such eigenvectors in H. However a generalization of the above statement holds in the form of the following Nuclear Spectral Theorem:[29)]

Let $\Phi \subset H \subset \Phi^X$ be a rigged Hilbert space and **A a cyclic** e.s.a. τ_Φ-continuous operator. Then there exists a set of generalized eigenvectors

(9) $F_\lambda = |\lambda\rangle \in \Phi^X$

(10) $A^X|\lambda\rangle = \lambda|\lambda\rangle, \lambda \in \Lambda$ i.e. $\langle A\phi|F_\lambda\rangle = \langle\phi|A^X F_\lambda\rangle = \lambda\langle\phi|F_\lambda\rangle$

for every $\phi \in \Phi$

such that for every $\phi, \psi \in \Phi$ ($\tilde{\phi} \in \Phi^{XX}$) and some uniquely defined positive measure μ on Λ

(11) $$(\psi,\tilde{\phi}) = \int_\Lambda d\mu(\lambda)\ \langle\psi|F_\lambda\rangle\langle F_\lambda|\tilde{\phi}\rangle = \int_\Lambda d\mu(\lambda)\langle\psi|\lambda\rangle\langle\lambda|\tilde{\phi}\rangle$$

which is written formally

(12) $$\tilde{\phi} = \int_\Lambda d\mu(\lambda)\ |\lambda\rangle\langle\lambda|\tilde{\phi}\rangle$$

If we set $\psi = \phi$ in (11) then we see that from the vanishing of all components $\langle\lambda|\tilde{\phi}\rangle$ of the spectral decomposition of ϕ corresponding to the operator A, it follows that $||\phi|| = 0$ i.e. that $\phi = 0$. Because of this property the set of generalized eigenvectors $|\lambda\rangle$ occurring in (11) or (12) is called complete (in analogy to the completeness of the system of ordinary eigenvectors in (8)).

This spectral theorem assures the existence of a complete set of generalized eigenvectors of the operator Q (and P), i.e. of generalized vectors that fulfill:

(13) $$Q^X|x\rangle = \bar{x}|x\rangle, \qquad P^X|p\rangle = \bar{p}|p\rangle$$

and with the help of which every $\phi \in \Phi$ can be written

(14) $$\tilde{\phi} = \int_X d\mu(x)\ |x\rangle\langle x|\phi\rangle, \quad \phi = \int_{\text{spectrum } P} d\mu(p)\ |p\rangle\langle p|\phi\rangle$$

where X =spectrum Q.

The fact that there is a complete set of eigenvectors does not tell us what this spectrum is.

It is widely known that the spectrum of Q and P is the real line. However it is not so widely known that the derivation[30] of this is far from trivial. The reason for this is that the problem of the physicist was the reverse of the problem described here, namely to find the defining assumptions (I,1) and (I,3) from the spectrum of Q which was assumed to be the real line. (cf. remark on p. 49).

We will in the following only describe the results, the derivation of which will be given in the Appendix V.

The set of generalized eigenvalues of Q and P is the complex plane. The spectrum of Q and P is the following subset of the set of generalized eigenvalues.

(15)
$$\text{spectrum } Q = \{x \mid -\infty < x < +\infty\}$$
$$\text{spectrum } P = \{p \mid -\infty < p < +\infty\}$$

In general, an e.s.a. operator has more generalized eigenvectors (and more generalized eigenvectors that are continuous functionals) than appear in the spectral decomposition (11), (12). (17)

The measure in (14) is the ordinary Lebesgue measure so that after normalization

$$\langle x' | x \rangle = \delta(x'-x)$$

we have

(16)
$$\tilde{\phi} = \int_{-\infty}^{+\infty} dx |x\rangle\langle x|\phi\rangle$$

Every vector $\tilde{\phi} \in \Phi$ is fully characterized by its components $\langle x|\tilde{\phi}\rangle$ with respect to the basis of generalized eigenvectors $|x\rangle$ (in the same way as every vector $\vec{x}$ in the 3-dimensional space can be equivalently described by its components x^i with respect to 3 basis vectors $\vec{e}_i$ according to $\vec{x} = \sum_{i=1}^{3} x^i \vec{e}_i$ or a vector $h \in H$ by its components (h_i,h) with respect to a complete basis system h_i $i=1,2,\ldots$). The function $\langle x|\tilde{\phi}\rangle$ is called a realization of the vector $\tilde{\phi}$ and the set of functions $\langle x|\tilde{\phi}\rangle$ for every $\tilde{\phi} \in \Phi$ is called a realization of the space Φ.

The realization of Φ is the space S of all infinitely differentiable functions $\phi(x) = \langle x|\phi\rangle$ which, together with their derivatives tend to zero for $|x|\to\infty$ more rapidly than any power of $\frac{1}{|x|}$ (Schwartz Space).

In this realization of Φ by S the operator Q is realized by the multiplication with x and the operator P is realized by the differential operator in S:

$$\langle x|Q|\phi\rangle = x \langle x|\phi\rangle$$

$$\langle x|P|\phi\rangle = \frac{\hbar}{i}\frac{d}{dx} \langle x|\phi\rangle$$

The elements of the algebra A are then realized by polynomials in x and $\frac{d}{dx}$.

We can, according to (III,12) consider $\tilde{\phi}$ in (14) or

(11) as the functional on the space Φ^X at the element $F \in \Phi^X$:

$$\tilde{\phi}(F) = \langle F|\tilde{\phi}\rangle = \overline{\langle\phi|F\rangle}$$

and write

$$\langle F|\tilde{\phi}\rangle = \int dx\, \langle F|x\rangle\langle x|\tilde{\phi}\rangle \tag{18}$$

The l.h.s. of (18) is well defined and, consequently, so is the r.h.s. However one factor in the product of the integrand, $\langle F|x\rangle$ has not been defined so far. We define $\langle F|x\rangle$ by (18). $\langle F|x\rangle$ is called a generalized function or distribution.

If the space Φ is realized by S, then the space Φ^X is realized by the space of all $\langle F|x\rangle$, $F \in \Phi^X$ defined by (18). This space S^X is called the space of tempered distributions.

In this realization the space H is realized by the space of square integrable functions $L^2(x)$. In distinction to the Hilbertspace formulation, in which the momentum operator $\bar{P}$ is realized by the differential operator on the subset of all differentiable functions $h(x)\in L^2(x)$ for which also $\frac{d}{dx}\,h(x)\in L^2(x)$, the momentum operator P in the present formulation is realized by the differential operator on S.

We remark, that in the derivation of these results (Appendix V) one has to use the defining assumption (I,3). This is a natural assumption for the harmonic oscillator, because H is the energy operator of the harmonic oscillator. For other physical, even if one assumes that its algebra be

generated by P_i, Q_i fulfilling the canonical commutation relation (I,1), the requirement that the operator $\Sigma P_i^2+Q_i^2$ be e.s.a. need not be such a natural assumption and therefore[33] (17) not the most natural possibility. Also it may well be that for other physical systems the algebra of observables may be more naturally defined by other operators than the P_i and Q_i.

Remark concerning generalization

For the case of a more general algebra of linear operators the e.s.a. operators are not cyclic (more than one quantum number is needed to characterize the pure states). One then needs a complete system of commuting operators to obtain a complete set of generalized eigenvectors.

$\{A_k\}$, $k = 1,2,\ldots N$ is a system of commuting operators iff

1.) $[A_i, A_k] = 0$ for all $i \neq k$

2.) $\sum_{k=1}^{N} A_k^2$ is e.s.a.

A_k is a complete commuting system iff there exists a vector $\phi \in \Phi$ such that

$$\{A\phi \mid A \in \text{algebra generated by } \{A_k\}\}$$

spans the space Φ(or H).

An antilinear functional F on Φ is a generalized eigenvector for the system A_k if for any $k = 1,\ldots N$

$$A_k^{\times} F = \bar{\lambda}^{(k)} F$$

The set of numbers $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots \lambda^{(N)})$ are called generalized eigenvalues corresponding to the generalized eigenvector $F_\lambda = |\lambda^{(1)} \lambda^{(2)} . \lambda^{(N)}>$. There exists a complete system of generalized eigenvectors according to the general form of the

Nuclear Spectral Theorem:

Let $\{A_k\}$ $k = 1,2,..N$ be a complete system of commuting e.s.a. τ_Φ-continuous operators in the rigged Hilbert space $\Phi \subset H \subset \Phi^X$

Then there exists a set of generalized eigenvectors

$|\lambda_1 .. \lambda_N> \in \Phi^X$

$$A_k^X |\lambda^{(1)} .. \lambda^{(N)}> = \lambda^{(k)} |\lambda^{(1)} .. \lambda^{(N)}>$$

$\lambda^{(k)} \in \Lambda^{(k)}$ = spectrum $\bar{A}_k$

such that for every $\phi \in \Phi$ and some uniquely defined measure μ on $\Lambda = \Lambda^{(1)} \times \Lambda^{(2)} .. \Lambda^{(N)}$

$$\tilde{\phi} = \int_\Lambda d\mu(\lambda) |\lambda^{(1)} .. \lambda^{(N)}><\lambda^{(1)} \ldots \lambda^{(N)}|\tilde{\phi}>$$

This theorem gives the precise formulation of the famous Dirac conjecture if the starting point is a precisely defined algebra.

The mathematical task that has to be solved if one starts with a well defined algebra is to find a complete com-

muting system and the spectrum of this complete commuting system. The problem to determine when a system is complete is far from trivial. Already for the simplest cases of enveloping algebras of group representations the number N is not independent of the particular commuting system, i.e. if $A_1 .. A_N$ is one particular complete commuting system then another commuting system containing an operator B_1 may require N+1 operators $B_1, B_2 ... B_{N+1}$ in order to be complete.[32)]

The problem of a physicist is usually the reverse of this mathematical problem. From the experimental data he finds out how many quantum numbers are required, and what are the possible values of these quantum numbers. This gives him the complete commuting system $\{A_k\}$ and its spectrum. He then conjectures the total algebra $\mathcal{A}$ by adjoining to $\{A_k\}$ a minimum number of other operators such that the matrix elements of elements of $\mathcal{A}$ calculated from the properties of this algebra agree with the experimental values of the corresponding observables. o

Summarizing: Using the rigged Hilbert space we have reproduced the main features of the Dirac formalism. For the particular case of the harmonic oscillator we have in fact reproduced everything of Diracs formalism. The difference between this formulation and the usual Hilbert space formulation appears to be minor from a physicists point of view but is essential from a mathematical point of view and leads

to tremendous mathematical simplification; in fact it justifies the mathematically undefined operations that the physicists have been accustomed to in their calculations.

APPENDIX I

LINEAR SPACES, LINEAR OPERATORS

A linear scalarproduct space Φ is (-not necessarily a space of functions but-) a set of mathematical (imagined) objects for which three operations are defined.

I Addition of elements of Φ, which has the following properties:

(i) For $h,f,g \in \Phi$ $\quad f + g \in \Phi$ and

$f + g = g + f$; $(f + g) + h = f + (g + h) =$

$f + g + h$

(ii) There exists an element $0 \in \Phi$ such that

$$f + 0 = f$$

(iii) $h + f = g$ has a unique solution $h \overset{def}{=} g - f$

II Multipliciation by complex numbers, which has the following properties

$\alpha(f + g) = \alpha f + \alpha g$, for $f,g \in \Phi$, $\alpha \in \mathcal{C}$ = complex numbers

$(\alpha + \beta)f = \alpha f + \beta f \qquad \alpha,\beta \in \mathcal{C}$

$\alpha(\beta f) = (\alpha\beta)f$; $0f = 0$

III Scalar product of two elements $\phi,\psi \in \Phi$ i.e. with every pair of elements $\phi,\psi \in \Phi$ there is associated a complex number (ϕ,ψ) with the properties

(i) $(\phi,\psi) = \overline{(\psi,\phi)}$

(ii) $(\alpha\phi_1 + \beta\phi_2,\psi) = \bar{\alpha}(\phi_1,\psi) + \bar{\beta}(\phi_2,\psi)$

(iii) $(\phi,\phi) \geq 0$ with $(\phi,\phi) = 0$ only for $\phi = 0$

An linear operator A is (-not necessarily a differential operation but-) a map from (a subset of) Φ into Φ which has the property:

$$A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Af_1 + \alpha_2 Af_2$$

$$\alpha_1,\alpha_2 \in \mathbb{C};\ f_1,f_2 \in \Phi$$

APPENDIX II

ALGEBRA

A set $\mathcal{A}$ is an (associative) algebra with unit element iff

(i) $\mathcal{A}$ is a linear space over $\mathcal{C}$.

(ii) For every pair $A,B \in \mathcal{A}$, there is defined a product $AB \in \mathcal{A}$ such that

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$(\alpha A)B = A(\alpha B) = \alpha\, AB$$

(iii) There exists an element $1 \in \mathcal{A}$ such that $1A = A$ for all $A \in \mathcal{A}$.

A set K of elements of $\mathcal{A}$ is called a system of generators iff the smallest closed subalgebra with unit which contains K coincides with $\mathcal{A}$. The element **1** is not to be called a generator.

Let the elements of K be called X_i $i = 1,2,..n$. Then each element of $\mathcal{A}$ can be written

$$(1)\quad A = C^o + \sum_{i=1}^{n} C^i X_i + \sum_{i,j}^{n} C^{ij} X_i X_j + \ldots.$$

$$C^o, C^i \ldots \in \mathcal{C}$$

We restrict ourselves to the case that n is finite and we will not discuss topologies of A, i.e. we assume that the above sums for every A are arbitrarily large but finite. Defining algebraic relations are relations among the generators

(2) $\quad P(X_i) = 0$

where $P(x_i)$ is a polynomial with complex coefficients of n variables x_i. An element $B \varepsilon A$

(3) $B = b^o + \Sigma\, b^i X_i + \Sigma\, b^{ij} X_i X_j + \ldots.$

$$b^i \ldots \varepsilon\, \mathcal{C}$$

is equal to the element A iff (3) can be brought into the same form (1) with the same coefficients $C^o, C^i, C^i \ldots.$ by the use of the defining relations (2).

The best known examples of associative algebras are the enveloping algebras $\varepsilon(G)$ of Lie groups G.

An enveloping algebra is the associative algebra generated by $X_1, X_2 \ldots X_n$ in which the multiplication is defined by the commutation relations

(4) $\quad P(X_i) = X_i X_j - X_j X_i - \sum_{k=1}^{n} C_{ij}{}^k X_k = 0$

$C_{ij}{}^k$ are called the structure constants of the Lie group G. The commutation relations (4) do in general not suffice to

fully define an enveloping algebra of linear operators in given linear topological spaces. I.e., there are several algebras of linear operators in linear topological spaces that fulfill a given commutation relation. In order to specify a particular algebra of operators one has to require additional algebraic relations and other properties (such as ΣX_i^2 be e.s.a.).

APPENDIX III

LINEAR OPERATORS IN HILBERT SPACES

We recall here a few of the definitions and important theorems from the theory of linear operators in Hilbert space.

Let $A:K \to K$ be a linear but not necessarily bounded operator on a Hilbert space K and let $D(A)$ be dense in K. Then A^*, the adjoint of A, is defined in K by

$$(A^*f,h) = (f,Ah).$$

The domain $D(A^*)$ is the set of all vectors $f \in K$ such that $(f,Ah) = (z,h)$ holds for all $h \in D(A)$; the vector z is then uniquely defined and $z = A^*f$.

An operator A on K is called symmetric if $D(A)$ is dense in K and $A \subset A^*$ (i.e., $D(A) \subset D(A^*)$ and $Af = A^*f$ for every $f \in D(A)$); it is called self-adjoint if $D(A)$ is dense in K and $A = A^*$.

An operator A on K is called closed if the relations

$$\lim_{n\to\infty} f_n = f, \quad \lim_{n\to\infty} Af_n = g, \quad f_n \in D(A)$$

imply $f \in D(A)$ and $Af = g$. Closedness is a weaker condition than continuity since, if an operator A on K is continuous, then

$$\lim_{n\to\infty} f_n = f \quad \text{for} \quad f_n \in D(A)$$

implies that the sequence $\{Af_n\}$ converges, while if it only closed, then the convergence of the sequence $\{f_n\}$ for $f_n \in D(A)$ does not imply the convergence of the sequence $\{Af_n\}$. However, if A is closed then, in particular, it has the property that two sequences $\{Af_n\}$ and $\{Ag_n\}$ cannot converge to different limits if the corresponding sequences $\{f_n\}$ and $\{g_n\}$ converge to the same limit. A* is always closed.

If A is not closed, it is sometimes possible to find an extension of A which is closed. An operator A admits of a closure $\bar{A}$ iff the relations

$$f_n \in D(A), \quad f'_n \in D(A), \qquad \lim_{n \to \infty} f_n = f, \qquad \lim_{n \to \infty} f'_n = f,$$

$$\lim Af_n = g, \ \lim Af'_n = g'$$

imply $g = g'$. In this case $D(\bar{A})$ consists of all $f \in K$ for which there exists a sequence $\{f_n\} \in D(A)$ which satisfies the conditions:

i) $\lim f_n = f$ and ii) $\{Af_n\}$ converges. Then by definition

$$\bar{A}f = \lim Af_n.$$

An operator A on K is called essentially self-adjoint if $\bar{A}$ is self-adjoint. Physical observable are assumed to be represented by essentially self-adjoint operators.

An operator A on K is called Hermitian if A is bounded and self-adjoint. An operator A on K is called unitary

if $||Af|| = ||f||$ for all $f \in K$. A unitary operator satisfies the relations $A^*A = AA^* = 1$.

Let M be a closed subspace of a Hilbert space K. An operator P which associates with each $f \in K$ its projection f_1 on M, $P:K \to M$ is called a projection operator on M. Projection operators are linear, Hermitian (i.e., $P^*=P$, $D(P)=K$) and idempotent (i.e., $P^2 = P$); and every linear, Hermitian, idempotent operator with $D(P) = K$ is a projection operator. Two projection operators P_1 and P_2 are called orthogonal iff $P_1P_2 = 0$; in this case P_1K and P_2K are orthogonal, i.e., for every $h_1 \in P_1K$ and every $h_2 \in P_2K$, we have $(h_1,h_2) = 0$.

APPENDIX IV:

PROOF OF LEMMA (14) SECTION II BY INDUCTION

(A1) From the c.r. follows: $(N+1)a^+ = a^+(N+2)$

$$a(N+1) = (N+2)a$$

(14) is true for p = 1 because

$$(\phi, a(N+1)a^+\phi) = (\phi, aa^+(N+2)\phi) = (\phi,(N+1)(N+2)\phi)$$

$$= (\phi,(N+1)^2\phi) + (\phi,(N+1)\phi) \leq 2(\phi,(N+1)^2\phi)$$

because $(\phi,(N+1)\phi) \leq (\phi_1(N+1)^2\phi)$

Assume (14) is true for p = q: i.e.

(A2) $(\phi, a(N+1)^q a^+\phi) \leq k(\phi,(N+1)^{q+1}\phi)$ for every ϕ

and calculate

$$(\phi, a(N+1)^{q+1}a^+\phi) = (\phi, a(N+1)(N+1)^{q-1}(N+1)a^+\phi) =$$

because of (A1)

$$=(\phi,(N+2)\, a(N+1)^{q-1}\, a^+(N+2)\phi)$$

$$\leq k((N+2)\phi, (N+1)^{q-1}(N+2)\phi)$$

because (A2) is valid for every ϕ, in particular for $\psi = (N+2)\phi$

$$\leq k[((N+1)\phi,(N+1)^{q-1}(N+1)\phi) + (\phi,(N+1)^q\phi) + ((N+1)\phi,(N+1)^{q-1}\phi)$$

$$+ (\phi,(N+1)^{q-1}\phi)]$$

$$\leq 4\cdot k\,(\phi,(N+1)^{q+1}\phi)$$

because of (6a).

Consequently (14) has been shown to be also fulfilled for

p = q+1 and therefore it is generally true.

APPENDIX V:

We will show in this Appendix that the defining assumptions, i.e. (I,1)(I,2)(I,3) and the τ_Φ-continuity of P and Q, will lead to the Schrödinger representation in the Schwartz-space S.[31) 33)]

We choose one of the energy eigenstates

Φ_n, n = 1,2,... and calculate (using $a\Phi_n = \sqrt{n}\ \Phi_{n-1}$, $a^+\Phi_n = \sqrt{n+1}\ \Phi_{n+1}$):

$$(1)\qquad \begin{array}{l} \langle Q\Phi_n|x\rangle = \frac{\hbar}{2m\omega}(\sqrt{n}\ \langle\Phi_{n-1}|x\rangle + \sqrt{n+1}\ \langle\Phi_{n+1}|x\rangle) \\ \quad \| \\ \bar{x}\langle\Phi_n|x\rangle \end{array}$$

for n = 0 one obtains

$$(1)\quad \sqrt{2}\ \frac{\bar{x}}{\sqrt{\hbar/m\omega}}\ \langle\Phi_o|x\rangle = \sqrt{1}\ \langle\Phi_1|x\rangle$$

(1) is a recurrence relation for $\langle\Phi_n|x\rangle$, which can be brought into a well known form by introducing

$$(2)\ y = \frac{\bar{x}}{\sqrt{\hbar/m\omega}}\ , f_n(y) = \sqrt{2^n n!}\ \frac{\langle\Phi_n|x\rangle}{\langle\Phi_o|x\rangle}\ (\text{assuming } \langle\Phi_o|x\rangle \neq 0)$$

(1) is then written

$$(1')\qquad f_{n+1}(y) = 2yf_n(y) - 2n\ f_{n-1}(y) \qquad n=1,2,\ldots$$

$$(1')\qquad f_1(y) = 2y\ f_o(y);\ f_o(y) = 1$$

(1') are the well known recurrence relations for the Hermite polynomials and have solutions for any complex y.

Thus we conclude that for any complex value x there is

an antilinear (not necessarily continuous) functional $|x\rangle = F_x$ which is a generalized eigenvector.

The functionals $\langle\Phi_n|x\rangle$ are given by

$$(3_x) \quad \langle\Phi_n|x\rangle = \frac{1}{\sqrt{2^n n!}} \langle\Phi_o|x\rangle \, H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right), \qquad x \in \mathbb{C}$$

As the operator Q is e.s.a. (as a consequence of (1,3)) the spectrum of Q must be real[28]. Thus the spectrum is only a subset of generalized eigenvalues and in the spectral decomposition

$$(4_x) \quad \Phi_n = \int d\mu(x) \; |x\rangle\langle x|\Phi_n\rangle$$

only those generalized eigenvectors and those functionals $\langle x|\Phi_n\rangle = \overline{\langle\Phi_n|x\rangle}$ appear for which x is real. To find the spectrum of Q then means to determine which real values x appear in (4_x).

If we consider Φ_n (or any $\phi \in \Phi$) as functional at the generalized eigenvector $F_{x'} = |x'\rangle \in \Phi^X$, $x' \in$ spectrum Q, then, according to (III,12), we obtain from (4_x):

$$(5_x) \quad \tilde{\Phi}_n(F_{x'}) = \langle x'|\Phi_n\rangle = \int d\mu(x) \; \langle x'|x\rangle\langle x|\Phi_n\rangle$$

Thus $d\mu(x) \langle x'|x\rangle$ must be the Dirac measure, i.e. the distribution $\langle x'|x\rangle$ defined by (5_x) must have the property of the Dirac δ- "functions" (where X is continuous):

$$(6_x) \quad d\mu(x) \, \langle x'|x\rangle = dx\,\delta(x'-x)$$

We consider now the generalized eigenvectors of the operator P:

$$P^{\times}|p\rangle = P^{\times} F_p = \bar{p} F_p$$

or $\langle P\phi|p\rangle = \bar{p}\langle\phi|p\rangle$ for every $\phi \in \Phi$.

Using $P = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(a-a^{+})$ one can proceed in complete analogy to the procedure for Q following eq. (1). One obtains that the set of generalized eigenvalues of P is the complex plane and

$$(3_p)\quad \langle\Phi_n|p\rangle = i^n \frac{1}{\sqrt{2^n n!}} \langle\Phi_o|p\rangle H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right)$$

In analogy to the case for Q, also in the spectral decomposition of Φ_n with respect to the operator P:

$$(4_p)\quad \Phi_n = \int d\mu(p)\ |p\rangle\langle p|\Phi_n\rangle$$

only those $\langle p|\Phi_n\rangle = \overline{\langle\Phi_n|p\rangle}$ of (3_p) occur for which p is real.

If we consider $\Phi_n \in \Phi$ as the functional on the space $\Phi^{\times}$ at the element $F_{p'} \in \Phi^{\times}$, then

$$(5_p)\quad \langle p'|\tilde{\phi}\rangle = \int d\mu(p)\ \langle p'|p\rangle\langle p|\tilde{\phi}\rangle$$

so that we conclude

$$(6_p)\quad d\mu(p)\ \langle p'|p\rangle = dp\ \delta(p'-p)$$

The analogy between the p- and x- spectral decomposition as expressed e.g. by the analogy between (3_x) and (3_p)

we should have expected as the assumptions we started with, (I,1) and (I,3) are symmetric in Q and P.

We now calculate the scalar product of Φ_n and Φ_m using (4_x) with (3_x) and (4_p) with (3p):

$$(7) \qquad \delta_{nm} = (\Phi_n,\Phi_m) = \int_X d\mu(x) \ \langle\Phi_n|x\rangle\langle x|\Phi_m\rangle$$

$$= \frac{1}{\sqrt{2^m m!}} \frac{1}{\sqrt{2^n n!}} \int_X d\mu(x) |\langle\Phi_o|x\rangle|^2 \ H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right) \overline{H_m\left(\frac{x}{\sqrt{\hbar/m\omega}}\right)}$$

$$= \int_{\text{spectrum } P} d\mu(p) \ \langle\Phi_n|p\rangle\langle p|\Phi_m\rangle$$

$$= \frac{1}{\sqrt{2^m m!}} \frac{1}{\sqrt{2^n n!}} \int_{(\text{spectrum } P)} d\mu(p) |\langle\Phi_o|p\rangle|^2 \ H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right) \overline{H_m\left(\frac{p}{\sqrt{\hbar m\omega}}\right)}$$

Comparing this with the orthogonality relations for the Hermite polynomials

$$(8) \quad \frac{1}{n!2^n\sqrt{\pi}} \int_{-\infty}^{+\infty} dy \ e^{-y^2} \ H_m(y) \ H_n(y) = \delta_{mn}$$

and taking into account that the Hermite polynomials are only orthogonal polynomials if associated with the interval $-\infty < y < +\infty$ and the weight e^{-y^2} (one can define $H_n(y)$ by (8) and derive (1') for real y) we conclude:

$$(9_x) \qquad d\mu(x)|\langle\Phi_o|x\rangle|^2 = \sqrt{\frac{m\omega}{\hbar\pi}} \ e^{-\frac{m\omega}{\hbar.2}x^2} dx$$

(10_x) spectrum $Q = X = \{x|-\infty<x<+\infty\}$

and

(9_p) $\quad d\mu(p)|\langle\Phi_o|p\rangle|^2 = \sqrt{\frac{1}{m\omega\hbar\ \pi}}\ e^{-\frac{p^2}{m\omega\hbar\cdot 2}}\ dp$

(10_p) $\quad$ spectrum $P = \{p|-\infty<p<+\infty\}$

If we agree to normalize the generalized eigenvectors such that

(11_x) $\quad \langle x'|x\rangle = \delta(x'-x)$

(11_p) $\quad \langle p'|p\rangle = \delta(p'-p)$

then according to (6_x) and (6_p)

(12_x) $\quad d\mu(x) = dx$

(12_p) $\quad d\mu(p) = dp$

and

(13_x) $\quad \langle\Phi_n|x\rangle = \sqrt[4]{\frac{m\omega}{\hbar}}\ \frac{1}{\sqrt{2^n n!\ \sqrt{\pi}}}\ H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right)\ e^{-\frac{m\omega}{\hbar 2}x^2}$

(13_p) $\quad \langle\Phi_n|p\rangle = i^n\ \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\ \sqrt[4]{\frac{1}{m\omega\hbar}}\ H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right)\ e^{-\frac{p^2}{m\omega\hbar.2}}$

Thus for we know the matrix elements of Q in the basis of generalized eigenvectors of Q

$(10'_x)$ $\langle x|Q|\phi\rangle = \overline{\langle Q\phi|x\rangle} = \overline{\langle\phi|Q^X|x\rangle} = x\langle x|\phi\rangle$

and the matrix elements of P in the basis of generalized eigenvectors of P

$(10'_p)$ $\langle p|P|\phi\rangle = p\,\langle p|\phi\rangle$

We now want to calculate the matrix elements of P in the basis of generalized eigenvectors of Q and the matrix elements of Q in the basis of generalized eigenvectors of P. In order to do this, we will consider Φ_n as a functional at the generalized eigenvector $F_p \varepsilon \Phi^X$, $p \varepsilon$ spectrum P, and use the spectral decomposition (4_x):

(14_x) $\langle p|\Phi_n\rangle = \int dx \langle p|x\rangle\langle x|\Phi_n\rangle$

and then as a functional at the generalized eigenvector $F_x \varepsilon \Phi^X$, $x \varepsilon$ Spectrum Q, and use the spectral decomposition (4_p):

(14_p) $\langle x|\Phi_n\rangle = \int dp \langle x|p\rangle\langle p|\Phi_n\rangle$

$\langle x|\Phi_n\rangle$ and $\langle p|\Phi_n\rangle$ in (14) are given by (13_x) and (13_p) respectively.

The Hermite polynomials have the following property:

$$(15)\qquad i^n e^{-\eta^2/2} H_n(\eta) = \int_{-\infty}^{+\infty} d\xi \frac{e^{i\xi\eta}}{\sqrt{2\pi}} e^{-\xi^2/2} H_n(\xi)$$

Inserting (13_p) and (13_x) into this relation it follows

$$(16_x)\qquad \langle\Phi_n|p\rangle = \int_{-\infty}^{+\infty} dx \frac{e^{\frac{ixp}{\hbar}}}{\sqrt{2\pi\hbar}} \langle\Phi_n|x\rangle$$

or taking the complex conjugate

$$(\overline{16_x}) \qquad \langle p|\Phi_n\rangle = \int dx \frac{e^{\frac{-ixp}{\hbar}}}{\sqrt{2\pi\hbar}} \langle x|\Phi_n\rangle$$

Comparing $(\overline{16_x})$ with (14_x) we find that the distributions $\langle p|x\rangle$ are given by:

$$(17_x) \qquad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{xp}{\hbar}}$$

In the same way one obtains from (15) and (14_p)

$$(17_p) \quad \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}$$

(17_x) and (17_p) together give

$$(18) \quad \langle x|p\rangle = \overline{\langle p|x\rangle} .$$

We emphasize that (18) does not follow from the "hermiticity property of the "scalar product" $\langle x|p\rangle$" but is a very particular property of the operators P and Q (and—as a consequence thereof—of the Fourier transformation).

For the general case of two arbitrary systems of generalized eigenvectors $|\lambda\rangle;\ A^X|\lambda\rangle = \bar{\lambda}|\lambda\rangle$

and $\qquad |\beta\rangle;\ B^X|\beta\rangle = \bar{\beta}|\beta\rangle$

it is <u>not</u> always true that

$$\langle\lambda|\beta\rangle = \overline{\langle\beta|\lambda\rangle}$$

though (III,12) holds always.

It is now simple to calculate the matrix element of P in the basis of generalized eigenvectors of Q, using (17_p)

$$\langle x|P|\phi\rangle = \int dp\ p\ \langle x|p\rangle\langle p|\phi\rangle$$

$$= \int dp\ \frac{\hbar}{i}\ \frac{d}{dx}\ \langle x|p\rangle\langle p|\phi\rangle$$

$$(19_x)\qquad \langle x|P|\phi\rangle = \frac{\hbar}{i}\ \frac{d}{dx}\ \langle x|\phi\rangle$$

In the same way using (17_x) one obtains

$$(19_p)\qquad \langle p|Q|\phi\rangle = -\frac{\hbar}{i}\ \frac{d}{dp}\ \langle p|\phi\rangle$$

It is now easy to see what the realization of Φ by the functions $\langle x|\phi\rangle$ (or $\langle p|\phi\rangle$) is: As every power Q^n and P^m of the operators P and Q are defined on Φ, the functions $\langle x|\phi\rangle = \phi(x)$ must fulfill:

$$(\phi, Q^n\tilde{\phi}) = \int dx\ x^n |\langle x|\tilde{\phi}\rangle|^2 < \infty$$

(20)

$$(\phi, P^m\tilde{\phi}) = (\frac{\hbar}{i})^m \int dx\ \overline{\langle x|\phi\rangle}\quad \frac{d^m}{dx^m}\ \langle x|\tilde{\phi}\rangle < \infty$$

$$(\phi, Q^nP^m\phi) = (\frac{\hbar}{i})^m \int dx\ \overline{\langle x|\phi\rangle}\ x^n\frac{d^m}{dx^n}\ \langle x|\tilde{\phi}\rangle < \infty$$

As all algebraic expressions in P and Q are continuous operators in Φ the sequence $\phi_\nu(x) = \langle x|\phi_\nu\rangle$, which is the realization of the convergent sequence $\phi_\nu \xrightarrow{\tau_\Phi} \phi$, converges to $\phi(x) = \langle x|\phi\rangle$ if the $x^n \frac{d}{dx^m}\phi_\nu(x)$ converge uniformly on every bounded region to $x^n \frac{d}{dx^m}\phi(x)$.

The space of functions $\phi(x)$ that fulfill (20) and for which the convergence is defined in the above described way

is the space of test functions S (Schwartz space).

Concluding this appendix we emphasize that in deriving (19) and (10) we have made use of (I,3), because by (I,3) we were led to (1) and therewith to the Hermite polynomials, whose properties have been used extensively to establish (19) and (10).

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7) The explanation of the content of this basic assumption is the subject of this paper. All the unknown mathematical notions will be defined or described in the following sections .... Appendices I(Linear Space), II(Algebra) and III(Operators in Hilbert Space) contain further definitions which are not given in the paper.

8) The formulation given here is a special case of the construction of Rigged Hilbert Spaces for enveloping algebras of Lie groups which has been suggested in A. Bohm Journ. Math. Phys. 8, 1557(1966), Appendix; and independently by B. Nagel, College de France Lectures (1970). A de- 16)
tailed treatment of $\varepsilon(SU(1,1))$, which is slightly more complicated than the example treated here, has been given in G. Lindblad, B. Nagel, Ann. Inst. Poincaré Section A-(N.S.) 13, 27(1970). 17)

9) See Appendix II

10) See Appendix I

11) See Appendix III 18)

12) E. Nelson, Ann. Math. 70, 572(1959).

13) H. D. Doebner Proceedings of the 1966 Istambul Summer Institute.

14) In a scalar product space the norm $|| \quad ||$ is defined by 19)
$||\phi|| = \sqrt{(\phi,\phi)}$.

15) The construction of the completion $\bar{R}$ of the space R consists of the following: The elements of the space $\bar{R}$ 20)
are all possible Cauchy sequences $x = \{x_n\}$, $x_n \in R$, where two such sequences $x = \{x_n\}$ and $y = \{y_n\}$ are not considered 21)
distinct if $||x_n - y_n|| \to 0$ and the elements $x \in R$ are identified in $\bar{R}$ with the sequence $\{x,x,x,\ldots..\}$. The

operations with these sequences are defined by

$$\alpha x = \{\alpha x_n\},\ x + y = \{x_n + y_n\},\ (x,y) = \lim_{n\to\infty} (x_n, y_n)$$

One can verify that $\bar{R}$ is then a complete scalar product space, i.e. a Hilbert space

That $(N+1)^p$ is e.s.a. can be proved in many ways. It also follows from the fact that $(N+1)^p$ is an elliptic element in the enveloping algebra of a group representation; see ref.20).

It is well known that P, Q and aa^+ cannot be continuous operators with respect to τ_H and are, therefore, not defined on the whole Hilbert space.

Let $\phi_\nu \xrightarrow{\tau_\phi} \phi$ i.e. $(\phi_\nu - \phi) \xrightarrow{\tau_\phi} 0$. Let us assume that A was not defined on ϕ. As A is a continuous operator $A(\phi_\nu - \phi) \xrightarrow{\tau_\phi} 0$. Therefore we can define A on ϕ by $A\phi = \tau_\phi\text{-}\lim A\phi_\nu$.

The definition of closed and closable operators is given in Appendix III. P,Q,H are closable because they are symmetric and a symmetric operator admits a closure.

E. Nelson, W. F. Stinespring, Amer. Journ. Math. 81, 547, (1959).

The original definition of nuclearity for countably normed spaces (Grothendieck, Gelfand and Kostyuchenko) is: Φ is nuclear iff for any m there is an n such that

the mapping $\Phi_n \to \Phi_m$ is nuclear i.e. has the form

$$\Phi_n \ni \phi \to \sum_{k=1}^{\infty} \lambda_k(\phi,\phi_k)_n \psi_k$$ where Φ_i is the completion

of Ψ with respect to the norm $\| \ \|_i$ and $\{\phi_k\},\{\psi_k\}$ are orthonormal systems in the spaces Φ_n and Φ_m respectively, $\lambda_k > 0$ and $\Sigma\lambda_k < \infty$.

22) J. E. Roberts Commun. Math. Phys. 3, 98(1966).

23) A. A. Kirillov, Dokl. Akad. Nauk. SSSR 130, 966(1960).

24) A. Bohm, Appendix B of Journ. Math. Phys. 8, 1557(1967).

25) B. Nagel, Lecture Notes, College de France (1970).

26) For a proof we refer to Gelfand, Shilov, ref. 1.) Vol. II, Ch. I, Sect. 5, 6 and Gelfand, Vilenkin, ref. 1.) Vol. IV, Ch. I, Sect. 3, 4.

27) This statement holds for countably Hilbert spaces if τ^X is the strong topology and in nuclear spaces—where strong and weak topology coincide— also for the weak topology. See ref. 25.

28) For a simple proof of this statement see e.g. Liusternik Sobolev, Elements of Functional Analysis §31 or Akhiezer, Glazman, Theory of Linear Operators in Hilbert Space §43.

29) For a proof of the Nuclear Spectral Theorem see ref. 1.) and ref. 2.)

30) J. Dixmier, Comp. Math. 13, 263, (1958) and references thereof.

31) See also, P. Kristensen, L. Meljbo, E. Thue Poulsen, Commun. Math. Phys. 1, 175, (1965).

32) G. Lindblad, B. Nagel, Ann. Inst. Poincaré Sect. A (N.S.) 13, 27(1970).

33) For the possibility of weakening assumption (I,3) see: L. C. Mejlbo, Math. Scand. 13, 129(1963); see also H. G. Tillmann, Act. Sci. Math. 24, 258(1963); C. Foias, L. Gehér, B. Sz-Nagy, Act. Sci. Math. 21, 78(1960).